Cotangent bundle : $T^*M = \coprod_{p \in M} T_p^*M$ = dual of T_pM.

$T_p^*M = \text{span}\{dx^i|_p\}$, dx^i = dual to $\frac{\partial}{\partial x^i}$.

Transition Rule : $(U_\alpha, \varphi_\alpha) \rightleftarrows (U_\beta, \varphi_\beta)$ different convention.

$\{x^i\}$ $\{y^\alpha\}$

$$dx^i = \sum_{\alpha=1}^{n} A_\alpha^i \, dy^\alpha$$

$$(dx^i)\left(\frac{\partial}{\partial y^\alpha}\right) = A_\alpha^i = dx^i\left(\sum_{j=1}^{n} \frac{\partial x^j}{\partial y^\alpha} \cdot \frac{\partial}{\partial x^j}\right)$$

$$= \sum_{j=1}^{n} \frac{\partial x^j}{\partial y^\alpha} \cdot \delta_j^i = \frac{\partial x^i}{\partial y^\alpha}$$

$$\boxed{\therefore \quad dx^i = \sum_{\alpha=1}^{n} \frac{\partial x^i}{\partial y^\alpha} dy^\alpha} \quad \text{V.S.} \quad \boxed{\frac{\partial}{\partial x^i} = \frac{\partial y^\alpha}{\partial x^i} \frac{\partial}{\partial y^\alpha}}$$

Last time : $TM = C^\infty$ mfd

Lemma : $T^*M = C^\infty$ mfd ~~~ talk about "covector field".

Sketch proof : suffices to find collection of chart on T^*M

which satisfies transition Rule.

chart of M

$$\Phi_\alpha : U_\alpha \times \mathbb{R}^n \longrightarrow \sum_{i=1}^{n} u_i \, dx^i \in T^*M$$

$$\Phi_\beta^{-1} \circ \Phi_\alpha (x, u) = (y^1(x), \cdots, y^n(x), \tilde{u}^1(x), \cdots, \tilde{u}^n(x))$$

i.e. $\displaystyle\sum_{\alpha=1}^n \tilde{u}^\alpha dy^\alpha = \sum_{i=1}^n u^i dx^i$, $\begin{cases} \Phi_\alpha \leftrightarrow (x^i) \\ \Phi_\beta \leftrightarrow (y^i) \end{cases}$

transition Rule //

$$\sum_{i=1}^n \sum_{\alpha=1}^n u^i \frac{\partial x^i}{\partial y^\alpha} dy^\alpha \Rightarrow \boxed{\tilde{u}^\alpha = u^i \boxed{\frac{\partial x^i}{\partial y^\alpha}}_{\in C^\infty} \quad \forall i, \alpha}$$

#

Example: Given $f: M \to \mathbb{R}$, smooth function on mfd M.

induce $df \in T^*M$ given by $\boxed{\boxed{df(X)} = X(f) \in C^\infty(M)}$.

$F: M^m \to N^n$ smooth map

$\rightsquigarrow$ induce $\underbrace{dF|_p}_{\substack{\| \\ \text{push forward}}} : \boxed{T_pM} \to \boxed{T_{F(p)}N}$ $\forall p \in M$

$(= F_*|_p)$

$\rightsquigarrow$ induce $F^*|_p : T^*_{F(p)}N \to T^*_pM$ by

let $\alpha \in T^*N$. $F^*\alpha$ be s.t. $\forall X \in TM$.

$$\underbrace{(F^*\alpha)}_{\in T^*M}(X) = \alpha((dF)X).$$

pull-back

generalize idea to multi-linear map!!

<u>tensor field</u> : given vector space V, W

$$\begin{matrix} V & W \\ \updownarrow & \updownarrow \\ V^* & W^* \end{matrix}$$

If $T \in V^*$, $S \in W^*$, we define

$T \otimes S : V \times W \to \mathbb{R}$ by

$$(T \otimes S)(a_1X_1 + a_2X_2, b_1Y_1 + b_2Y_2)$$

$$\stackrel{\Delta}{=}$$

$$a_1b_1\, T(X_1)S(Y_1) + a_1b_2\, T(X_1)S(Y_2) + a_2b_1\, T(X_2)S(Y_1) + a_2b_2\, T(X_2)S(Y_2)$$

<u>Rmk</u>: Define $T_1 \otimes T_2 \otimes T_3 \otimes \cdots \otimes T_k$ inductively.

$V^* \otimes W^* = \text{span}\{ T \otimes S : T \in V^*, S \in W^* \}$ = vector space.

On mfd. $\forall p \in M$, $V = T_pM$, $W = T_pM$

$V \otimes V = T_pM \otimes T_pM$. pointwise construction

$TM = \coprod_{p \in M} T_pM$.

$T_pM \times T_pM \to \mathbb{R}$ (pointwise level)

$TM \otimes TM = \{ T : TM \times TM \to C^\infty(M) \}$

k 2-copies k

<u>Defn</u>: • A tensor T of order $(\cancel{2}\,k, 0)$ on mfd is a multi-linear mapping over $C^\infty(M)$.

• A tensor T of order $(k, 1)$ on mfd is a multi-linear mapping over $C^\infty(M)$ s.t.

$$T : \overbrace{TM \times TM \times \cdots \times TM}^{k\text{-copies}} \to TM.$$

(if $k=1$, tensor of order $(1,1)$ on mfd $\in End(TM)$)
if T is invertible.

Locally, tensor T of order $(k,0)$, is in form of

$$T = \sum T_{i_1 i_2 i_3 \cdots i_k} \, dx^{i_1} \otimes dx^{i_2} \otimes \cdots \otimes dx^{i_k}$$

where $T_{i_1 i_2 \cdots i_k}$ are locally smooth fcn.

Riemannian Metrics :

Defn : $g \in T^*M \otimes T^*M$ is a metric on M if

① g is positive definite $\longleftrightarrow$ $g(X,X) \geq 0 \quad \forall X \in TM$ and "=" iff $X=0$.

② g is symmetric $\longleftrightarrow$ $g(X,Y) = g(Y,X) \quad \forall X, Y \in TM$

③ g is C^∞ $\longleftrightarrow$ $g = g_{ij} \, dx^i \otimes dx^j$ where $g_{ij} \in C^\infty_{loc}$.

Defn : $F: (M,g) \to (N,h)$ is a diffeomorphism then

F = Isometry iff $F^* h = g$. ✲✲ stronger than Isometry as metric space

---

Example : ① $(\mathbb{R}^n, \delta)$, $\delta = \sum_{i=1}^{n} dx_i \otimes dx^i$ = flat.

② (H^n, g) where $g = \sum 4 \dfrac{dx^i \otimes dx^i}{(1-|x|^2)^2}$. $H^n = \{|x|<1\} \subseteq \mathbb{R}^n$

③ $\Sigma^n \subseteq \mathbb{R}^{n+1}$ hypersurface

$g = g_{ij} \, dx^i \otimes dx^j$ given by 1st fundamental form
coordinate of Σ, Not $\mathbb{R}^{n+1}$.

④ (S^2, g), $g = d\varphi^2 + \sin^2\varphi \, d\theta^2$ = Round sphere.

|| special case in ③.

Rmk: $F : M^m \to (N^n, h)$ C^∞ map

$\rightsquigarrow F^*h \in T^*M \otimes T^*M$ not necessarily a metric on M since

For $X, Y \in TM$, $(F^*h)(X,Y) = h(F_*X, F_*Y)$ (might be $= 0$) ≥ 0

If F = immersion, then $dF \neq 0$ ← Rule out

$\Rightarrow F^*h$ = Riemannian metric on M.

Q: Are smooth mfd admitting metric g??

Ans: Yes if M is Hausdorff w/ countable base.

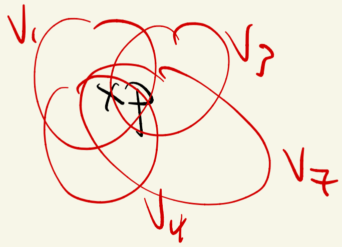
pf: $\boxed{\text{cheating proof: By Whitney embedding thm, } \exists\, F : M^n \hookrightarrow \mathbb{R}^{2n} \text{, embedding; define metric on } M \text{ by } F^* g_{\mathbb{R}^{2n}}}$

Alternative pf:

preparation (partition of unity)

Let $\{V_\alpha\}_{\alpha=1}^\infty$ be collection of open sets s.t. $\forall p \in M$,

$\exists$ open $W \ni p$ s.t. $|\{\alpha : W \cap V_\alpha \neq \emptyset\}| < +\infty$



coordination

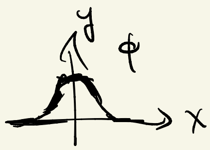
And Refine V_α s.t. $\varphi_\alpha(V_\alpha) \subseteq B(0,1) \subseteq \mathbb{R}^n$

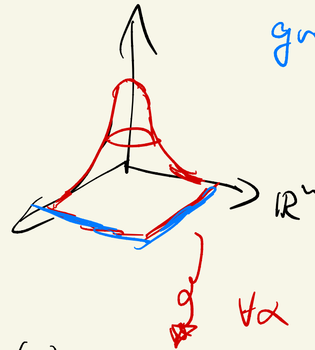
graph of f

Define: $f_\alpha : B(1) \to \mathbb{R}$ as

$\quad\;\; \| $

$\quad \phi(|x|)$





(sketch only)

take look on J.M. Lee Smooth mfd

$\forall x \in M$, define $h_\alpha(x) = \dfrac{f_\alpha(x)}{\sum_\beta f_\beta(x)} \geq 0$, smooth

= finite sum for each x

$$\Rightarrow \sum_{\alpha=1}^{\infty} h_\alpha(x) = 1 \text{ on } M.$$

partition of unity.

---

on M, define $g|_p(u,v) = \sum_{\alpha=1}^{\infty} h_\alpha(p)\cdot\langle u,v\rangle_{\mathbb{R}^n}$.

$\| $

metric.

Riemannian metric $\rightsquigarrow$ volume / measure

- measure element which is indep. of chart.

$d\mu = F(x)\, dx^1 \cdots dx^n$ (locally on $\{x^i\}$)

$\qquad \|$

$\;\; = G(y)\, dy^1 \cdots dy^n$ (on $\{y^\alpha\}$)

$$F(x)\, dx^1 \cdots dx^n = \underbrace{F(x(y))}_{G(y)} \cdot \det\left[\frac{\partial x^i}{\partial y^\alpha}\right] dy^1 \cdots dy^n$$

Goal

achieve

Observe: if $F(x) = \sqrt{\det g_{ij}}$ where $g_{ij} = g\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right)$

then

$$\boxed{g_{ij} = g\left(\frac{\partial y^\alpha}{\partial x^i}\frac{\partial}{\partial y^\alpha}, \frac{\partial y^\beta}{\partial x^j}\frac{\partial}{\partial y^\beta}\right) = \frac{\partial y^\alpha}{\partial x^i}\frac{\partial y^\beta}{\partial x^j}\, g_{\alpha\beta}}$$

$$\underbrace{\det g_{ij}}_{= F(x)^2} = \underbrace{\det g_{\alpha\beta}}_{= G(y)^2} \cdot \left[\det\left(\frac{\partial y^\alpha}{\partial x^i}\right)\right]^2$$

$$\Longrightarrow \quad F(x)\, dx^1 \cdots dx^n = G(y)\, dy^1 \cdots dy^n \quad \text{if} \quad \det\left(\frac{\partial y^\alpha}{\partial x^i}\right) > 0$$

Def.: If $\exists\ \{(U_\alpha, \varphi_\alpha)\}$ s.t. $\forall\ \alpha, i$, $\det\left(\frac{\partial y^\alpha}{\partial x^i}\right) > 0$,

then M = orientable.

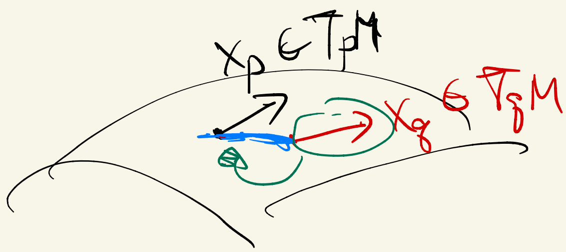
In this case, $d\mu \triangleq \sqrt{\det g_{ij}}\, dx^1 \cdots dx^n$ define a measure on M which is indep. of charts.

$$\mathrm{Vol}(\Omega) = \int_\Omega d\mu = \int_{\varphi_\alpha^{-1}(\Omega)} \sqrt{\det g}\, dx^1 \cdots dx^n \quad \text{if } \Omega \subseteq \varphi_\alpha(U_\alpha)$$

$$\int_\Omega f \, d\mu = \sum_{\alpha=1}^{\infty} \int_\Omega f \, h_\alpha \, d\mu \quad \text{for general measurable function } f: M \to \mathbb{R}.$$

$\mathcal{L}_X Y$: Lie-derivative depending on C^∞ structure.

Goal: Define some derivative of vector field compatible with g !!



Q: How to move $X_q \in T_qM$ to $? \in T_pM$ which "agree" with g.

Def: An Affine connection := bi-linear map

$$\nabla : \Gamma(TM) \times \Gamma(TM) \longrightarrow \Gamma(TM) \quad \text{s.t.}$$

① $\nabla_{fV} W = f \nabla_V W$

② $\nabla_V (fW) = V(f) \cdot W + f \nabla_V W \quad \forall f \in C^\infty(M)$.

③ $\nabla_V (W + Z) = \nabla_V W + \nabla_V Z$

④ $\nabla_{V+W} Z = \nabla_V Z + \nabla_W Z$.

$\nabla_\cdot \cdot \, / \, \mathcal{L}_\cdot \cdot$ $\Gamma(TM)$ $\rightsquigarrow$ $\Gamma(TM)$

For $X, Y, Z \in \Gamma(TM)$, $g(X,Y) \in C^\infty(M)$

$$Z\big(g(X,Y)\big) = \underbrace{\boxed{Z(g)}(X,Y)}_{??\,\|} + g(\cancel{Z(X)}^{\nabla_Z X}, Y) + g(X, \cancel{Z(Y)}^{\nabla_Z Y})$$

"derivative of g wrt Z. $\to 0$

Defn: The Affine connection ∇ is said to be compatible with g if $\forall X, Y, Z \in \Gamma(TM)$,

$$Z\big(g(X,Y)\big) = g(\nabla_Z X, Y) + g(X, \nabla_Z Y)$$

Thm: Let (M, g) be a Riemannian mfd, then $\exists!\, \nabla$ on M s.t.

Levi-Civita Connection ∇

① ∇ is compatible with g

② ∇ is torsion free i.e. $\nabla_X Y - \nabla_Y X = [X, Y]$ $\forall X, Y \in \Gamma(TM)$.

only depends on C^∞ structure

Pf: if ∇ exists, then

$$\begin{cases} X\langle Y, Z\rangle = \langle \nabla_X Y, Z\rangle + \langle Y, \nabla_X Z\rangle \\ +Y\langle X, Z\rangle = \langle \nabla_Y Z, X\rangle + \langle Z, \nabla_Y X\rangle \\ -Z\langle X, Y\rangle = -\langle \nabla_Z X, Y\rangle - \langle X, \nabla_Z Y\rangle \end{cases}, \forall X, Y, Z.$$

$$= \langle X, [Y, Z]\rangle + \langle Y, [X, Z]\rangle$$

$$+ \langle Z, [X, Y]\rangle + 2\langle Z, \nabla_Y X\rangle$$

$\therefore$ $\forall X, Y, Z$, we have

$$2\langle Z, \nabla_Y X\rangle = \Big(X\langle Y, Z\rangle + Y\langle X, Z\rangle - Z\langle X, Y\rangle\Big)$$

$$\underbrace{- \Big(\langle X, [Y, Z]\rangle + \langle Y, [X, Z]\rangle + \langle Z, [X, Y]\rangle\Big)}_{\text{only depends on Lie Bracket}}$$

$\Rightarrow$ $\nabla_X Y$ = uniquely determined by g

Locally: $\left\langle \nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j}, \frac{\partial}{\partial x^k} \right\rangle = \frac{1}{2}\left(\partial_j g_{ik} + \partial_i g_{jk} - \partial_k g_{ij}\right)$, $\forall i, j, k$

$$\nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j} \overset{\Delta}{=} \Gamma^p_{ij} \frac{\partial}{\partial x^p}$$

$$\Rightarrow g^{kl}\, \Gamma^p_{ij}\, g_{pk} = \frac{1}{2}\left(\partial_j g_{ik} + \partial_i g_{jk} - \partial_k g_{ij}\right) g^{kl}$$

$\boxed{g^{kl} = \text{inverse of } g \quad (\text{i.e. } g^{ij} g_{jk} = \delta^i_k)}$

∇ determ on $\left\{ \begin{array}{l} \Gamma^l_{ij} = \frac{1}{2} g^{lk}\left(\partial_i g_{jk} + \partial_j g_{ik} - \partial_k g_{ij}\right) \\ \Gamma^l_{ij} = \Gamma^l_{ji} \quad (\text{from } \nabla_i \partial_j = \nabla_j \partial_i) \end{array}\right.$ #

Check: If we define ∇ s.t. $\nabla_i \partial_j = \Gamma_{ij}^k \partial_k$, $\Gamma_{ij}^k = \Gamma_{ji}^k$

and with the "bilinear" properties,

then ∇ is compatible with g. (Ex.) #

☆☆ Importance Concept in RG: tensor

Example: $T(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y]$ [if ∇ = some general affine connect)

is a tensor.

- $T(fX,Y) = \nabla_{fX} Y - \nabla_Y(fX) - [fX,Y]$

$$= f\nabla_X Y - \cancel{Y(f)\cdot X} - f\nabla_Y X$$

$$- (fXY - \cancel{Y(f)\cdot X} - fYX)$$

$$= f(\nabla_X Y - \nabla_Y X - [X,Y]) = fT(X,Y)$$

$\forall f \in C^\infty(M)$.

- linear over $C^\infty(M)$

- $T(X,fY) = fT(X,Y)$ By symmetry

Lemma: Let $X, Y \in \Gamma(TM)$, $\tilde{X}, \tilde{Y} \in \Gamma(TM)$ s.t. $\boxed{\begin{cases} X(p) = \tilde{X}(p) \\ Y(p) = \tilde{Y}(p) \end{cases}}$

$\left(Q: T(X,Y) \overset{?}{=} T(\tilde{X},\tilde{Y}) \right)$

then $\alpha T(X,Y) = \alpha T(\tilde{X},\tilde{Y})$ at p. $\forall \alpha \in T^*M \otimes T^*M$.

pf: Suffices to show that "if $X(p)=0$, then $T(X,Y)\big|_p = 0$".

let $X \in \Gamma(TM)$ s.t. $X(p) = 0$.

then $X = x^i \partial_i$ where $\begin{cases} x^i(\cdot) \text{ is smooth} \\ x^i(p) = 0 \end{cases}$.

$$\alpha T(X,Y)\big|_p = \alpha T(x^i \partial_i, Y)\big|_p = \overset{0}{\cancel{x^i\big|_p}}\, \alpha T(\partial_i, Y)\big|_p = 0.$$

$\therefore \alpha T(X,Y)$ only depends on $X(p)$, $Y(p)$. ☆

RmK: • $\nabla_X Y \neq$ tensorial on Y
(depending on extension of Y)

• $\nabla_X Y =$ tensorial on X.
(indep. of extension of X)